Solutions to selected exercises

from

Gravity from the ground up

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Gravity from the ground up

Solutions to selected exercises

Chapter 1

Exercise 1.1.1: ***Speed of a falling body*** [page 3]

Using the fact that the acceleration of gravity on Earth is $g = 9.8\,\mathrm{m\,s^{-2}}$, calculate the speed a ball would have after falling for two seconds, if dropped from rest. Calculate its speed if it were thrown downwards with an initial speed of $10\,\mathrm{m\,s^{-1}}$. Calculate its speed if it were initially thrown *upwards* with a speed of $10\,\mathrm{m\,s^{-1}}$. Is it falling or still rising after 2 s?

Solution 1.1.1

When the ball falls from rest, its speed is given by Equation 1.3 with $v_0 = 0$:

$$-9.8\,\mathrm{m\,s^{-2}} \times 2\,\mathrm{s} = -19.6\,\mathrm{m\,s^{-1}}.$$

We take negative quantities to indicate the downward direction, so g and the speed are negative in this calculation. When the initial speed is $v_0 = -10\,\mathrm{m\,s^{-1}}$ (negative because it is downwards), we add this to the first result to get $-29.6\,\mathrm{m\,s^{-1}}$. When the ball is thrown upwards, the initial speed of $v_0 = 10\,\mathrm{m\,s^{-1}}$ reduces the speed acquired from gravity; the result is $-9.6\,\mathrm{m\,s^{-1}}$. This is still negative, so the ball is falling down after two seconds. If the initial speed upwards had been larger, say $30\,\mathrm{m\,s^{-1}}$, then after two seconds the ball would still be rising at a speed of $(30 - 19.6)\,\mathrm{m\,s^{-1}}$, or $10.4\,\mathrm{m\,s^{-1}}$.

Exercise 1.2.1: ***Distance fallen by a body*** [page 4]

For the falling ball in Exercise 1.1.1, calculate the distance the ball falls in each of the cases posed in that exercise.

Solution 1.2.1

When the ball falls from rest, its distance is given by Equation 1.6 with v_0 and d_0 set to zero:

$$\frac{1}{2} \times \left(-9.8\,\mathrm{m\,s^{2}}\right) \times 4\,\mathrm{s}^2 = -19.6\,\mathrm{m}.$$

Again, we take negative quantities to indicate the downward direction, so g and the distance traveled are negative in this calculation. To this basic number we must add the term $v_0 t$ for the different situations. (We can take d_0 to be zero, since the problem asks how far the ball falls from its release point.) In the second case, the initial speed of $-10\,\mathrm{m\,s^{-1}}$ (downwards) adds $-10\,\mathrm{m\,s^{-1}} \times 2\,\mathrm{s} = -20\,\mathrm{m}$ to the distance, giving a fall of $-39.6\,\mathrm{m}$. When the ball is thrown upwards, we add a positive $20\,\mathrm{m}$ to the basic distance, giving a net displacement of $0.4\,\mathrm{m}$. This is positive, which means that the ball has not quite fallen back to its initial release height after two seconds.

Exercise 1.3.1: *Small steps in speed and distance* [page 5]

Suppose that at the n^{th} time-step t_n, the vertical speed is v_n and the vertical distance above the ground is h_n. Show that at the next time-step $t_{n+1} = t_n + \Delta t$, the vertical speed is $v_{n+1} = v_n - g\Delta t$. Using our method of approximating the distance traveled by using the average speed over the interval, show that at the next time-step the height will be

$$h_{n+1} = h_n + {}^1\!/\!_2(v_n + v_{n+1})\Delta t = h_n + v_n \Delta t - {}^1\!/\!_2 g(\Delta t)^2.$$

Solution 1.3.1

Since the vertical speed decreases by $g\Delta t$ in each time-interval Δt, the expression for v_{n+1} is straightforward. The first expression for h_{n+1} is based on our approximation that *distance = average speed × time.* The average speed is, of course, $\frac{1}{2}(v_n + v_{n+1})$. The second expression is the result of substituting our original answer for v_{n+1} into this.

Exercise 1.3.2: *Suicide shot* [page 5]

What is the *minimum* range of a cannonball fired with a given speed V, and at what angle should it be aimed in order to achieve this minimum?

Solution 1.3.2

Zero: fire it vertically. Then run away.

Exercise 1.3.3: *Maximum range by algebra* [page 5]

For readers interested in verifying the guess we made above from the numerical data, here is how to calculate the range at 45° algebraically. The range is limited by the amount of time the cannonball stays in the air. Fired at 45° with speed V, how long does it take to reach its maximum height, which is where its vertical speed goes to zero? Then how long does it take to return to the ground? What is the total time in the air? How far does it go horizontally during this time? This is the maximum range.

Solution 1.3.3

At 45°, the initial vertical speed is $V/\sqrt{2}$. To reach a speed of zero, it must ascend for a time t_{up} such that $V/\sqrt{2} = gt_{\text{up}}$. (We take g positive here.) This means $t_{\text{up}} = V/(g\sqrt{2})$. The time it takes the cannonball to fall back down is the same, so that the total time is $t_{\text{tot}} = 2t_{\text{up}} = \sqrt{2}V/g$. During all this time, the cannonball has been travelling with a horizontal speed of $V/\sqrt{2}$ as well. This means it travels a total distance $d = t_{\text{tot}} V/\sqrt{2} = V^2/g$. This is what we inferred from the computer results.

Exercise 1.3.4: *Best angle of fire* [page 5]

Prove that 45° is the firing angle that gives the longest range by calculating the range for any angle and then finding what angle makes it a maximum. Use the same method as in Exercise 1.3.3 on page 5.

Solution 1.3.4

Let the angle of fire be θ, as measured up from the ground as in Figure 1.2. The initial vertical speed is then $V\sin\theta$, and by the argument in the solution to Aexrefaex:maxrange, the time spent in the air is $t_{\text{tot}} = 2V\sin\theta/g$. The horizontal speed is $V\cos\theta$, and so the distance travelled is then $d = \frac{2V^2}{g}\sin\theta\cos\theta$. By a standard trigonometric identity, the product $2\sin\theta\cos\theta$ is $\sin 2\theta$. The maximum of the sine function occurs when its argument is 90°, and in our case the argument is 2θ. Therefore, the maximum range occurs for an angle $\theta = 90°/2 = 45°$.

Gravity from the ground up
Solutions to selected exercises
Chapter 2

Exercise 2.2.1: ***Redshift to a satellite*** [page 16]

Calculate the redshift gh/c^2 if h is the distance from the ground to a satellite in low-Earth orbit, 300 km. Suppose the "light" is actually a radio wave with a frequency of 10^{11} Hz. How many cycles would the transmitter emit if it ran for one day? How many fewer would be received in one day by the satellite? How long did it take the transmitter to generate these "extra" cycles?

Solution 2.2.1

The redshift is 3.3×10^{-11}. One day is 86400 s, during which the transmitter emits 10^{11} cycles per second of radio waves, leading to a total of 8.64×10^{15} cycles. Because the redshift is the fractional amount by which the frequency is reduced, the receiver counts $3.3 \times 10^{-11} \times 8.64 \times 10^{15} = 2.85 \times 10^{5}$ fewer cycles in the same amount of time at the receiver, leading to the conclusion that the transmitter is running slower than the receiver. The transmitter generated these cycles in $2.85 \times 10^{5}/(10^{11}\,\mathrm{Hz} = 3\,\mu\mathrm{s}$. These calculations really only apply to a "satellite" that is *at rest* at the height of 300 km; the orbital motion of a real satellite introduces other Doppler effects and effects due to special relativity that are discussed in Chapter 15.

Gravity from the ground up

Solutions to selected exercises

Chapter 3

Exercise 3.1.1: *Vectors* [page 22]

Quantities that have a value but no direction are called **scalars**. Decide whether the following physical quantities should be described mathematically by scalars or by vectors: (a) the mass of a rock; (b) the electric force on a charged particle; (c) the temperature of a room; (d) the slope of a hill.

Solution 3.1.1

(i) The mass of a rock is a pure number, with no directional information; it is a scalar. *(ii)* The electric force on a charged particle has a direction as well as a size, so it must be described by a vector. *(iii)* The temperature of a room again has no directional information; for example, it is the same no matter what direction one holds the thermomenter. So it is a scalar. *(iv)* The slope of a hill can be described by a vector pointing in the direction of steepest ascent, whose length is proportional to the steepness, say equal to the height the slope rises in a unit horizontal distance.

Exercise 3.1.2: *Period of a satellite orbiting near the Earth's surface* [page 22]

Use Equation 3.1 on page 19 to calculate the orbital period of a satellite near the Earth. Assume that the acceleration of gravity at the height of the satellite is the same as on the ground, $g = 9.8\,\mathrm{m\,s^{-2}}$. Take the radius of the orbit to be the radius of the Earth, 6400 km, plus the height of the satellite above the Earth, 300 km.

Solution 3.1.2

In every calculation, it is important to check that one is using the right units. In this case, be careful to express the radius of the orbit in meters, not the given value of km. The orbit is given as 6700 km, which is $6.7 \times 10^6\,\mathrm{m}$. Now, multiplying g by R gives $9.8\,\mathrm{m\,s^{-2}} \times 6.7 \times 10^6\,\mathrm{m} = 6.57 \times 10^7\,\mathrm{m^2\,s^s}-2$. (Notice how I have kept the units in the expression, multiplying them in the same way as the numbers.) Equation 3.1 on page 19 tells us to take the square root of this to get the orbital speed: $V = 8.1 \times 10^3\,\mathrm{m\,s^{-1}}$, where again the units have been "square-rooted" too. The circumference of the orbit is $2\pi R = 4.21 \times 10^7\,\mathrm{m}$, so the orbital period is this circumference divided by the speed: $P = 5.2 \times 10^3\,\mathrm{s}$. Dividing by 60 to convert this to minutes gives 86.6 min. The velocity quoted in the text is slightly smaller ($7.9\,\mathrm{km\,s^{-1}}$), and the period slightly longer (90 min), because in fact gravity is slighly weaker at the height of the satellite.

Gravity from the ground up
Solutions to selected exercises

Chapter 4

Exercise 4.1.1: *Inverse-square-law constant* [page 29]

From Equation 4.2 on page 27, show that the constant k in Equation 4.4 on page 29 is $(2\pi)^2/(P^2/R^3)$. Evaluate this to give $1.327 \times 10^{11}\,\mathrm{km^3\,s^{-2}}$.

Solution 4.1.1

Solve Equation 4.2 on page 27 for a:

$$a = \frac{4\pi^2}{R^2}\left(\frac{P^2}{R^3}\right)^{-1}.$$

Then solve Equation 4.4 on page 29 for k: $k = R^2 a$. Then put our new expression for a into this to get

$$k = 4\pi^2\left(\frac{P^2}{R^3}\right)^{-1}.$$

The value of P^2/R^3 from Table 4.2 on page 28 can then be used to obtain k.

Exercise 4.1.2: *Measuring the mass of the Sun and the Earth* [page 29]

The Newtonian law of gravity, Equation 4.1 on page 27, tells us the force on a body of mass M_2 exerted by the Sun (mass M_1 in the equation). Combine this with Newton's second law, $F = ma$, to show that the acceleration of the body of mass M_2 is $a = GM_1/r^2$. Use this to show that the force-law constant k in Equation 4.4 on page 29 is $k = GM_1$. Convert this value of k to more conventional units using meters to find $k = 1.327 \times 10^{20}\,\mathrm{m^3\,s^{-2}}$. (Hint: since $1\,\mathrm{km} = 10^3\,\mathrm{m}$, it follows that $1 = 10^3\,\mathrm{m\,km^{-1}}$. Multiply by the cube of this form of the number 1 to convert the units for k.) Now use the value of $G = 6.6725 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}$ to find the mass of the Sun. Do a similar calculation for the value of Kepler's constant for the Moon, given in Table 4.2 on page 28, to find the mass of the Earth.

Solution 4.1.2

The motion of mass M_2 is governed by the equation $F = M_2 a$, from which we have $a = F/M_2$. Since the force of gravity is $F = GM_1M_2/r^2$, we find $a = GM_1/r^2$. By comparison with Equation 4.4 on page 29, which says that $a = k/r^2$, we see that $k = GM_1$. Since our value of k uses units of km and s, which are convenient for the computer program but not conventional, and since the conventional units for G involve m, s, and kg, we need to convert units. Unit conversions are always done by multiplying by some version of the number 1. In our case, if we start from

$$1\,\mathrm{km} = 10^3\,\mathrm{m},$$

and divide by the left-hand-side, we get (multiplying and dividing units algebraically)

$$1 = 10^3\,\mathrm{m\,km^{-1}}.$$

We can multiply the cube of this into our value of k to get

$$\begin{aligned} k &= 1.327 \times 10^{11}\,\mathrm{km^3\,s^{-2}} \quad \times \quad 1^3 \\ &= 1.327 \times 10^{11}\,\mathrm{km^3\,s^{-2}} \quad \times \quad 10^9\,\mathrm{m^3\,km^{-3}} \\ &= 1.327 \times 10^{20}\,\mathrm{m^3\,s^{-2}}, \end{aligned}$$

where again we have cancelled units algebraically. Since $k = GM_1$, it follows that $M_1 = k/G$, so that

$$M_1 = 1.327 \times 10^{20}\,\mathrm{m^3\,s^{-2}} \quad \times \quad \left(6.6725 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}\right),$$

which gives 1.989×10^{30} kg. For the Earth, the value of k, inferred as above from the Moon's orbit as given in Table 4.2 on page 28 and then converted to conventional units, is $k = 4.012 \times 10^{14}\,\mathrm{m^3\,s^{-2}}$. Dividing by G gives the Earth's mass, 6.01×10^{24} kg. This is close to but not actually equal to the true mass of the Earth, because the numbers we use assume a circular orbit for the Moon. But the closeness of our answer to the right one shows how the method is used.

Exercise 4.3.1: *Area of Colorado* [page 35]

The American state of Colorado is a spherical rectangle of the kind we have just described. Its northern and southern boundaries have latitude 41° and 37°, respectively. Its eastern and western boundaries have longitude 102° and approximately 109.1°, respectively. Given that the radius of the Earth is 6.3782×10^6 m, what is the area of Colorado?

Solution 4.3.1

The difference in latitude angles is 4° and the difference in longitudes is 7.1°. The cosine of the latitude in the center of the state is $\cos 39° = 0.77715$. Then Equation 4.11 on page 35 gives an area of 2.7×10^{11} m^2. This is of course a first approximation to the area, since the formula works only for small rectangles over which the cosine factor does not change much. We can test this by asking what the cosine is at the northern boundary: $\cos 41° = 0.75471$. This differs by about 3% from the value at the center, so we can expect that our value for the area of Colorado is accurate at about the 3% level. There are other approximations involved here as well. For one thing, Colorado is not at sea level; its mean elevation is between 1 and 2 km higher, so we should have used a larger radius. This correction is less than 0.1%, however. A second additional correction is that Colorado is not a smooth spherical patch: we have neglected the roughness of Colorado due to its mountains! We can't estimate this correction without more data on the topography of the mountains, but it is probably not as large as our first uncertainty of 3%, because the mountains are typically less than 2 km above the mean elevation of the state.

Exercise 4.4.1: *Light deflection by other bodies* [page 38]

Any gravitating body will deflect light. Estimate, using Equation 4.13 on page 38 above, the amount of deflection experienced by a light ray just grazing the surface of the following bodies: (a) Jupiter, whose radius is 7.1×10^4 km; (b) the Earth; (c) a black hole of any mass; and (d) you.

Solution 4.4.1

We use Equation 4.13 on page 38. We need to know M and d in each case. (a) Getting the mass of Jupiter from Table 4.2 on page 28 and using the given balue of R gives a deflection angle of 4×10^{-8} radians, or 8 milliarcsec (1 milliarcsec is 10^{-3} arcseconds).

(b) For the Earth we get 1.4×10^{-9} radians, or 0.3 milliarcsec.

(c) A black hole has a radius $R_g = 2GM/c^2$, so it follows that the deflection angle $2GM/c^2d$ with $d = R_g$ always evaluates to 1 radians, or 57°, regardless of the mass of the black hole. However, this is well beyond the small-angle approximation, so all we can say is that the deflection will be large. In fact, light that gets near to the horizon can circle around the hole many times before getting away again, leading to deflection angles much larger than 360°.

(d) To make yourself approximate a spherical gravitating body, imaging crouching into a ball of radius about $R = 0.5\,\mathrm{m}$. Assuming your mass is $50\,\mathrm{kg}$, we find that the deflection is 1.5×10^{-25} radians, or about 3×10^{-20} arcsec.

Gravity from the ground up

Solutions to selected exercises

Chapter 5

Exercise 5.1.1: ***Testing the binomial approximation*** [page 43]

Use a pocket calculator to verify that Equation 5.2 on page 43 gives a good approximation for small ϵ. For the following values of ϵ and n, evaluate the approximate value $1 + n\epsilon$, the exact value $(1 + \epsilon)^n$, the error (their difference) and the *relative error* of the approximation, which is defined as the error divided by the exact value: (a) $n = 2$, $\epsilon = 0.01$, 0.1, 1.0; (b) $n = 3.5$, $\epsilon = 0.01$, 0.1; and (c) $n = -2$, $\epsilon = 0.01$, 0.1. (Recall that negative powers indicate the reciprocal, so that $(1 + \epsilon)^{-2} = 1/(1 + \epsilon)^2$.)

Solution 5.1.1

In the table of answers below, the error is defined as $(1+\epsilon)^n - (1+n\epsilon)$. The relative error is the error divided by $(1 + \epsilon)^n$.

n	ϵ	Exact	Approximate	Error	Relative error
2	0.01	1.0201	1.02	0.0001	9.8×10^{-5}
2	0.1	1.21	1.2	0.01	0.0083
2	1.0	4	3	1	0.25
3.5	0.01	1.0354397	1.03	0.0054397	0.00525351
3.5	0.1	1.3959646	1.35	0.04596458	0.0329268
-2	0.01	0.980296	0.98	0.000296	0.000302
-2	0.1	0.8264463	0.8	0.026446	0.032

The relative error is about 1% or less for most of these values. The approximation is clearly terrible for $\epsilon = 1$, but of course our derivation would lead us to expect this.

Gravity from the ground up

Solutions to selected exercises

Chapter 6

Exercise 6.1.1: ***Escaping from anywhere*** [page 53]
Calculate the escape speed from the Solar System for a satellite starting at the average distance from the Sun of each of the planets listed in Table 4.2 on page 28. In each case, find the ratio of this speed to the average speed of the planet (column 5 of the table).

Solution 6.1.1
For the escape speed we use the formula $v_{\rm escape} = (2GM_\odot/r)^{1/2}$, with values of G and $M_\odot$ taken from the Appendix. Then we divide the result by the tabulated average speed to get the ratio of escape speed to orbital speed. The results are in the following table. All speeds are in km s^{-1}.

Planet:	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
$v_{\rm escape}$:	67.5	49.7	42.1	34.2	18.5	13.7	9.63	7.69	6.72
Ratio:	1.41	1.41	1.41	1.42	1.41	1.42	1.42	1.42	1.42

The last row of the table shows that the escape speeds are all close to the orbital speeds times $\sqrt{2}$, as would be expected for perfectly circular orbits. The small differences are due to the small eccentricities of the true orbits.

Exercise 6.2.1: ***Solving the quadratic equation*** [page 58]
The general solution of the quadratic equation $ax^2 + bx + c = 0$ for x is

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\left(b^2 - 4ac\right)^{1/2}, \qquad (6.17)$$

where the $\pm$ sign indicates that there are two solutions, found by taking either sign in the expression. Apply this formula to solve the quadratic equation above for R_2. Show that the two roots are R_1 and the root given by Equation 6.14 on page 58.

Solution 6.2.1
Comparing the form $ax^2 + bx + c = 0$ with the equation

$$\left(\frac{2L_1}{R_1} - 1\right) R_2^2 - 2L_1R_2 + R_1^2 = 0,$$

we make the identifications $x = R_2$, $a = (2L_1/R_1) - 1$, $b = -2L_1$, and $c = {R_1}^2$. The crucial expression $b^2 - 4ac$ then becomes $4{L_1}^2 - 8L_1R_1 + 4{R_1}^2$, which is the exact square of $2L_1 - 2R_1$. Therefore, we can do the square root explicitly to get from Equation 6.17 on page 58 (after cancelling a common factor of 2)

$$R_2 = \frac{1}{a}\left[L_1 \pm (L_1 - R_1)\right].$$

Taking $1/a = R_1/(2L_1 - R_1)$, we get for the plus sign $R_2 = R_1$ (the "trivial" root, which of course must come out as a solution), and for the minus sign $R_2 = {R_1}^2/(2L_1 - R_1)$, as given in Equation 6.14 on page 58.

Exercise 6.2.2: ***Getting from the Earth to other planets*** [page 58]

Use Equation 6.16 on page 58 to calculate the speed needed to go from the Earth's orbit to the orbits of Mars, Jupiter, and Saturn. The derivation of this formula actually did not need to assume that $r > 1$, so use it for Venus, too.

Solution 6.2.2

The values of r needed for the different planets are just their distances from the Sun in AU, column 2 of Table 4.2 on page 28. The results are: Mars, $32.7\,\mathrm{km\,s^{-1}}$; Jupiter, $38.6\,\mathrm{km\,s^{-1}}$; Saturn, $40.1\,\mathrm{km\,s^{-1}}$; Venus, $27.3\,\mathrm{km\,s^{-1}}$.

Gravity from the ground up

Solutions to selected exercises

Chapter 7

Exercise 7.1.1: ***How fast does a helium balloon rise?*** [page 73]

The density of the helium in a balloon filled at, say, a fairground is $0.18\,\mathrm{kg\,m^{-3}}$, while the density of the air around it is $1.3\,\mathrm{kg\,m^{-3}}$. Using Equation 7.1 on page 73, compute the pressure difference across the air that the balloon will displace, assuming for simplicity that it is a *cube* of side 20 cm. Then compute from this the net pressure force on the balloon itself when it is inflated and takes the place of the air. Next, compute the weight of the balloon (neglecting the rubber of the balloon itself) and calculate its initial acceleration (upwards). What multiple of g is this? Will it keep this acceleration as it moves upwards? How many balloons would be required to lift a 60 kg woman?

Solution 7.1.1

We take the balloon to be cubical to avoid complications arising from averaging the forces around a sphere. If the balloon will have a height h of 0.2 m, then Equation 7.1 on page 73 gives (with $g = 9.8\,\mathrm{m\,s^{-2}}$ and the density of *air* $\rho = 1.3\,\mathrm{kg\,m^{-3}}$) $\Delta p = 2.5\,\mathrm{N\,m^{-2}}$. This is the pressure difference that supports the cube of air that occupies the volume that the balloon will fill when inflated. When the balloon inflates, it displaces the air, but it feels the same pressure difference, since this comes from the air around the balloon, which hardly notices the replacement.

The cube has a surface area of $0.04\,\mathrm{m^2}$ top and bottom, so the net pressure force across the balloon (as across the original cube of air) is 0.10 N. This is what we call the buoyancy force on the balloon. It is upwardly directed, because the pressure at the top is less than the pressure at the bottom.

The weight of the balloon is F_{gravity} as calculated in Investigation 7.1 on page 73: the density of helium times the volume of the balloon, $0.008\,\mathrm{m^3}$, times g, which works out to 0.014 N. This is a force pulling down on the balloon, so it subtracts from the buoyancy force of the air to give a *net upward force* of about 0.09 N.

The acceleration of the balloon upwards is this force divided by the balloon's mass, which is its density times its volume. The result is $a = 60\,\mathrm{m\,s^{-2}}$. Dividing by g, we find that this is about $6g$. This is the initial acceleration of the balloon, but the acceleration is soon reduced by friction with the air it moves through.

To lift the woman requires an upward force larger than her weight, $mg = 60\,\mathrm{kg} \times 9.8\,\mathrm{m\,s^{-2}} = 588\,\mathrm{N}$. If each balloon can exert an upward force of 0.09 N, then we need $588/0.09 \approx 6500$ balloons! So helium-balloon sellers at fairs are safely anchored to the ground by their own weight, and cartoons that show small children lofted into the air by such balloons are, of course, just cartoons.

Readers who are comfortable with algebra can save themselves a large amount of arithmetic — and learn more about the problem — by using symbols for the various quantities until the very end. The pressure difference is $\Delta p = -g\rho_{\text{air}}h$, the net pressure force is $F_{\text{pressure}} = g\rho_{\text{air}}h^3$ (positive because upwards), and the gravitational force is $F_{\text{gravity}} = -g\rho_{\text{helium}}h^3$. The net force on the balloon is $F_{\text{net}} = g(\rho_{\text{air}} - \rho_{\text{helium}})h^3$, so the acceleration is $a = F_{\text{net}}/(\rho_{\text{helium}}h^3) = g(\rho_{\text{air}}/\rho_{\text{helium}} - 1)$. The multiple of g is therefore $\rho_{\text{air}}/\rho_{\text{helium}} - 1$. This evaluates to $6g$ as above.

Even more important than the saving in calculation is the fact that we learn other things from the algebraic approach to this problem. In particular, we see that the result we are looking for does not depend on the *size* of the balloon h: all the arithmetic using h above eventually cancelled out in the end. This also suggests that the result would not depend on the *shape* of the balloon, so our assumption of a cubical balloon is not so unrealistic after all!

Exercise 7.2.1: *How many atoms in a balloon?* [page 78]

Consider the cubical helium-filled balloon of Exercise 7.1.1 on page 73. If the pressure inside the balloon is atmospheric pressure, $p = 10^5\,\mathrm{N\,m^{-2}}$, and the temperature is $T = 300\,\mathrm{K}$ (about 81 F), then use Equation 7.6 on page 78 to calculate the number N of helium atoms in the balloon. The size of this answer justifies the approximation that we can average over large numbers of randomly moving atoms.

Solution 7.2.1

The ideal gas law Equation 7.6 on page 78 can be solved for N to give $N = pV/(kT)$. Using the given values for all these numbers (the volume comes from the fact that the balloon is a cube of side $0.2\,\mathrm{m}$), we find that $N = 2 \times 10^{23}$ atoms.

Exercise 7.2.2: *What is the mass of a helium atom?* [page 78]

Use the answer to the previous exercise and the density of helium given in Exercise 7.1.1 on page 73 to calculate the mass of each helium atom. Use the density given for air to calculate the *average* mass of an air molecule. (Since air is a mixture of gases, we only obtain the *average* mass this way.)

Solution 7.2.2

The density is the mass per unit volume. By dividing our previous result by the volume of the cube, we get the number of atoms per unit volume, often called the number density. This is $2.5 \times 10^{25}\,\mathrm{m^{-3}}$. (The units mean "atoms per cubic meter". There is no explicit indication of "atoms" in the units because the number of atoms is a pure number, without dimensions.) The mass of each atom is the mass density divided by the number density, or $7 \times 10^{-27}\,\mathrm{kg}$. For air, the number density is the same, since the ideal gas law does not require us to specify what the gas is made of. So if we use the given mass density for air we find that the average molecular mass is $5 \times 10^{-26}\,\mathrm{kg}$. We see from this that the mean mass of an air molecule is 7 times larger than that of a helium atom, which is another way to understand why helium rises.

Gravity from the ground up

Solutions to selected exercises

Chapter 8

Exercise 8.1.1: *Frequency of light* [page 87]

Find the frequency (in Hz) of light whose wavelength is $0.5\,\mu$m.

Solution 8.1.1

Using Equation 8.3 on page 87, we get $f = (2.998 \times 10^8\,\mathrm{m\,s^{-1}})/(5 \times 10^{-7}\,\mathrm{m}) \approx 6 \times 10^{14}$ *unittime*−1, or 6×10^{14} Hz. Visible light oscillates more than 10^{14} times per second. (Compare this to audible sound frequencies, where the air pressure oscillates no faster than about 2×10^4 times per second.)

Exercise 8.1.2: *Photons from a light-bulb* [page 87]

Show that a 100 W light bulb (which emits 100 J of energy each second) must be giving off something like 10^{21} photons per second.

Solution 8.1.2

If each photon carries, by Equation 8.5 on page 87, an energy of about 10^{-19} J, then to emit 100 J requires 10^{21} photons. Not all of these necessarily come out visible light (whose energy would be somewhat larger than 10^{-19} J): tungsten filament lightbulbs produce only a few percent of their energy output in the visible region of the spectrum. The rest comes out as "heat radiation", which is infrared and has wavelengths longer than $1\,\mu$m.

Exercise 8.1.3: *Sunburn* [page 87]

The DNA molecules that carry genetic information in the nuclei of living cells are very sensitive to light with a wavelength of $0.26\,\mu$m, which breaks up DNA molecules. Deduce from this the binding energy of the chemical bonds within DNA. Ultraviolet light of wavelength $0.28\,\mu$m is the most effective for inducing sunburn. What is the threshold energy required to stimulate the chemical reactions that lead to sunburn?

Solution 8.1.3

Using $0.26\,\mu$m for λ in Equation 8.5 on page 87, we obtain an energy $E = 7.7 \times 10^{-19}$ J for the bonds within DNA. For light of wavelength $0.28\,\mu$m, we have $E = 7.1 \times 10^{-19}$ J. This must be close to the threshold for stimulating damaging sunburn reactions, since light of longer wavelengths and hence lower energies does not do so.

Exercise 8.1.4: *Gamma-rays* [page 87]

When some elementary particles decay, they give off so-called **gamma-rays**, which are really high-energy photons. A typical energy released in this way is 10^{-12} J. What is the wavelength of such a gamma-ray? What is its frequency?

Solution 8.1.4

Using Equation 8.5 on page 87, setting $E = 10^{-12}\,\mathrm{J}$, and solving for λ gives $\lambda = 2\times 10^{-6}\,\mathrm{m}\times(10^{-19}\,\mathrm{J}/10^{-12}\,\mathrm{J}) = 2\times 10^{-13}\,\mathrm{m}$.

Exercise 8.2.1: *Temperature of the Sun* [page 89]

If one analyzes the colors of the Sun, one finds that the greatest amount of light is emitted in the *blue–green* region of the spectrum, around $0.5\,\mu$m. Show that this gives an *estimate* of the Sun's temperature of $T = 19\,000\,\mathrm{K}$. This is closer to the real temperature (5600 K) than our estimate in the text, but we will get a much better estimate by refining this technique in the next chapter. (The eye sees the Sun as yellow, not blue–green, partly because it has greater sensitivity to yellow light and partly because blue light is scattered by the atmosphere.)

Solution 8.2.1

By using Equation 8.8 on page 89 with $\lambda = 0.5\,\mu$m, and then solving for T, we get $T = 0.6\times 9600\,\mathrm{K} \approx 5800\,\mathrm{K}$.

Gravity from the ground up

Solutions to selected exercises

Chapter 9

Exercise 9.2.1: ***Magnitude of the Sun*** [page 108]

Use the solar constant, given just before Equation 9.1 on page 106, to compute the apparent magnitude m of the Sun, using Equation 9.2 on page 108. Use Equation 9.4 on page 108 to calculate the absolute magnitude M of the Sun.

Solution 9.2.1

The solar constant (flux of energy from the Sun at the Earth's position) is $1355\,\mathrm{W\,m^{-2}}$. The apparent magnitude then comes out at $m_\odot = -26.9$. The absolute magnitude of the Sun depends on its luminosity, given by Equation 9.1 on page 106. Using this in Equation 9.4 on page 108 gives $M_\odot = 4.83$.

Exercise 9.2.2: ***Stellar magnitudes*** [page 108]

A particular star is known to be ten times further away than αCen and five times more luminous. Compute its apparent and absolute magnitudes.

Solution 9.2.2

The flux of light from the star is directly proportional to its luminosity, but inversely proportional to the *square* of its distance. The flux from the given star is therefore $5/100 = 0.05$ times that of αCen. The apparent magnitude depends on -2.5 times the logarithm of the ratios of fluxes, so that the apparent magnitude of the given star equals that of αCen plus $-2.5\log(0.05) = 3.25$. Since the apparent magnitude of αCen is 0.08, as given in the text, the apparent magnitude of the given star is 3.33. Its absolute magnitude depends only on its luminosity, not its distance from us. Since it is five times more luminous than αCen, its absolute magnitude is that of αCen plus $-2.5\log(5) = -1.7$. The absolute magnitude of αCen is 4.3, so that of the given star is 2.6.

Gravity from the ground up

Solutions to selected exercises

Chapter 11

Exercise 11.1.1: *Chemical bangs* [page 123]

Can it really be true that chemical reactions all give off the same energy per atom, to within a factor of, say, 10 or 100? Don't the chemical reactions that make a TNT bomb explode give off far more energy than the chemical reactions that heat up a smoldering rubbish dump? Explain why the answer to this question is no.

Solution 11.1.1

Chemical reactions differ tremendously in their *rate*, that is the number of reactions that take place in a given time. The trick of making an explosive is to make sure that all the atoms participate in the reactions, and that the reactions happen fast, so that all the energy is released in a very short time. The total released, per atom taking part in the reactions, is not very different from one substance to another.

Exercise 11.1.2: *Turning on the lights* [page 123]

Once a cloud of gas begins to contract to form a star, roughly how long does it take before nuclear reactions begin to power the star? Will the star shine before this?

Solution 11.1.2

The cloud of gas starts out very diffuse, so it must contract by a large factor to form a star. This contraction releases the sort of gravitational energy we have just discussed, and so the star can indeed shine. But we have seen that gravitational energy can last only a few million years, so that is the time after which nuclear reactions must take over.

Exercise 11.2.1: *Water power* [page 124]

If all the hydrogen in a teaspoonful of water were converted into helium, how long would that water power a 100 W light bulb? Take a teaspoon to contain 5 g of water.

Solution 11.2.1

Water molecules consist of one oxygen atom, containing 32 protons and neutrons, and two hydrogen atoms, with one proton each. So hydrogen makes up about 5.9% of the weight of the water. (Electrons have negligible mass for this problem.) There is thus 0.29 g of hydrogen in the teaspoon. Since a proton has a mass of 1.67×10^{-24} g, there are 1.8×10^{23} hydrogen protons in the spoon. When four of them combine to form helium, they release 5×10^{-29} kg converted into energy. Therefore the whole spoon will yield 2×10^{-6} kg converted into energy. Multiplying by c^2 to find the equivalent energy gives 2×10^{11} J. The 100 W light bulb uses 100 J per second, so the spoonful will power the bulb for 2×10^{9} s, or about 65 years.

Exercise 11.5.1: ***Entropy of the Sun*** [page 133]

We have seen in Chapter 8 that a typical photon in the Sun takes 10^6 y to randomly "walk" out of the Sun. That means that the Sun contains all the photon energy it generated by nuclear reactions in the last million years. This must all be in the form of photons, since the particles in the Sun have the same total energy today as they had a million years ago. (a) Calculate from the solar luminosity how much energy the Sun contains in photons. (b) If the average temperature inside the Sun is 10^5 K, calculate mean energy of each photon. (c) From these two results estimate the number of photons inside the Sun. (d) From the mass of the Sun, assuming for simplicity that it is composed entirely of hydrogen, calculate the number of protons (hydrogen nuclei) in the Sun. (e) Find the ratio of the number of photons to the number of protons in the Sun. This is a measure of what physicists call the **entropy** of the Sun.

Solution 11.5.1

(a) For a luminosity of 3.83×10^{26} W, the total energy emitted during the last million years (3×10^{13} s) is 1.2×10^{40} J. This is roughly the amount of energy that is currently contained in the photon gas inside the Sun.

(b) The typical photon energy is the same as that of the particles with which the photons are constantly colliding, namely $E = {}^3\!/_2 kT$. For a temperature of 10^5 K, the energy is 2×10^{-18} J.

(c) This means that the number of photons is $(1.2 \times 10^{40}\ \mathrm{J})/(2 \times 10^{-18}\ \mathrm{J}) = 6 \times 10^{57}$.

(d) We want to compare this with the number of protons in the Sun, assuming that all the mass of the Sun (2×10^{30} kg) is in protons, each of which has a mass 1.67×10^{-27} kg. (This is a good assumption, since most of the mass of the Sun is pure hydrogen, and the mass in electrons is negligible.) Dividing one by the other gives 1.2×10^{57} protons.

(e) There are thus five times as many photons inside the Sun as protons.

Gravity from the ground up

Solutions to selected exercises

Chapter 12

Exercise 12.1.1: ***The conditions for star formation*** [page 137]

A typical molecular cloud has a temperature $T = 20\,\mathrm{K}$, a composition mainly of molecules of H_2 (molecular hydrogen), and a density that corresponds to having only 10^9 molecules of H_2 per cubic meter. Calculate the Jeans length and Jeans mass of this cloud. Compare the mass you get with the mass of the Sun.

Solution 12.1.1

For the given data, the Jeans length is $\lambda_J = 3.4 \times 10^{16}\,\mathrm{m}$. In astronomers' distance units, this is 1.1 pc, fairly typical of the distances between stars. The Jeans mass, which is the mass within a sphere of this size in the molecular cloud, is $5.5 \times 10^{32}\,\mathrm{kg}$, or 280 solar masses. This means that the region of a cloud that will contract to begin forming stars will typically start to form hundreds of stars.

Exercise 12.2.1: ***Inverting a logarithmic equation*** [page 141]

Go through the steps leading from Equation 12.5 to Equation 12.6 on page 141. First put the definition of β into Equation 12.5. Then write $m \log M$ as $\log(M^m)$. Finally combine this term with the β term on the right-hand side of Equation 12.5 on page 141 to get the logarithm of Equation 12.6 on page 141. Justify each step you make in terms of the rules given above for the use of logarithms.

Solution 12.2.1

We use the definition $b = \log \beta$ to write Equation 12.5 on page 141 as

$$\log L = m \log M + \log \beta.$$

Then we use the identity $a \log x = \log(x^a)$ to write this as

$$\log L = \log(M^m) + \log \beta.$$

Next we use the identity $\log x + \log y = \log(xy)$ to write this as

$$\log L = \log(\beta M^m).$$

Finally we raise both sides to the power 10, using $10^{\log x} = x$, to get

$$L = \beta M^m.$$

since in the discussion we decided that the slope m was about 3.5, this is the same as Equation 12.6 on page 141.

Exercise 12.2.2: ***Dependence of stellar lifetime and luminosity on mass*** [page 141]

In the same way as we estimated the exponent in the relationship between luminosity and mass, estimate the exponent in the relationship between lifetime and mass. Do you get −2.5, as given above? The curves for luminosity and lifetime are not perfect straight lines, so representing them by a single constant exponent is an approximation. Estimate the error in this approximation by giving a range of values for both exponents that would acceptably represent the graphs.

Solution 12.2.2

When the mass of the star is 1 solar mass, the lifetime is about 2×10^4 million years. When the mass of the star increases to 30 solar masses, the lifetime decreases to about 5 million years. In terms of logarithms, the logarithm of the lifetime has decreased by 3.6 while the logarithm of the mass has increased by about 1.5. The slope is the ratio of these numbers, about −2.4. (The minus sign occurs because the lifetime has decreased, not increased.) This is close to the value of −2.5 that is usually taken as the rule of thumb. If we look at the graph in Figure 12.3 on page 139, we see that the slope is a bit steeper than our average value over the lower range of masses, only getting shallow at the higher end. Slopes between −3 and −2 would probably fit different parts of the curve.

Exercise 12.2.3: ***Energy radiated per kilogram: is it a constant?*** [page 141]

Test the assertion that the energy radiated per kilogram is constant, independent of the mass, by estimating the exponent from the graph in the same way as the lifetime and luminosity exponents were estimated. Is the exponent really zero? If not, can you explain this in terms of the uncertainty in the other two exponents that you arrived at in the previous exercise? In other words, is the exponent you get for the radiated energy within the range of exponents you would get if you selected various values for the other exponents and put them into the relation in Equation 12.7 on page 141?

Solution 12.2.3

At one solar mass the energy consumption is about 2 million joules per kilogram, rising to about 5 at 30 solar masses. The logarithmic slope is about 0.3. This is fairly close to zero, and would be well within the uncertainties of 0.5 in the slopes of the other curves.

Exercise 12.3.1: ***Boson stars*** [page 143]

(a) In Equation 12.11 on page 143, replace the proton and electron masses by a single boson mass m_{b} and assume that $\mu = 1$. This gives the formula for a star composed of just one type a particle, the boson of mass m_{b}. For such a star of total mass M, calculate the following ratio

$$2GM/Rc^2 = 8M^2 m_{\mathrm{b}}^2/m_{\mathrm{Pl}}^4,$$

where m_{Pl} is the Planck mass defined in Equation 12.20 on page 146.

(b) The ratio above is the ratio of the size of a black hole, $2GM/c^2$, as given in Equation 4.12 on page 36, to the size of the star. We will see in Chapter 21 that the star cannot be smaller than a black hole, so this ratio must be less than one. Show that this sets a maximum mass on a boson star made from bosons of mass m_{b}:

$$M_{\max} = 8^{-1/2} m_{\mathrm{Pl}}^2/m_{\mathrm{b}}.$$

(c) Find the largest mass m_{b} that the boson could have in order to allow boson stars of a solar mass to exist. Find the ratio of this mass to the mass of a proton. You should find that the boson needs to have very much less mass than a proton.

Solution 12.3.1

(a) With the given assumptions, the boson star has a radius $R = h^2/4Gm_{\mathrm{b}}{}^2M$. To see how relativistic it is, we form the ratio $2GM/Rc^2 = 8(Gm_{\mathrm{b}}M/hc)^2$. If we now use Equation 12.20 on page 146 to replace the combination G/hc by $1/m_{\mathrm{Pl}}$, we get the desired expression, $8m_{\mathrm{b}}{}^2M^2/m_{\mathrm{Pl}}{}^4$.

(b) Since $8m_{\mathrm{b}}{}^2M^2/m_{\mathrm{Pl}}{}^4$ cannot exceed 1, it follows that M cannot exceed $8^{-1/2}m_{\mathrm{Pl}}{}^2/m_{\mathrm{b}}$.

(c) For boson stars of a solar mass to exist, the maximum mass given above must be at least $M_\odot$. This means that m_b must be smaller than ${m_{Pl}}^2/M_\odot$. Using the value of 5.5×10^{-8} kg given for the Planck mass in Equation 12.20 on page 146, we find that the boson mass must be less than 1.5×10^{-45} kg. This is only roughly a fraction 10^{-20} of the proton mass!

Such a small-mass particle could easily have avoided discovery in particle-physics experiments so far, if it does not interact with ordinary matter.

Exercise 12.4.1: *Momentum in the Fermi sea* [page 144]

Here is where the factor of $N_e^{1/3}$ comes from. First we consider the easier case of electrons confined in a one-dimensional "box", say along a string of finite length. We return to the three-dimensional star later. If each pair of electrons has a distinct momentum, separated by $\Delta m_e v$ from its neighbors, then we could mark out a line on a piece of paper, start with the smallest momentum allowed ($\Delta m_e v$), and make a mark each step of $\Delta m_e v$. Each mark represents the momentum of one pair of electrons. If we have N_e electrons, then there will be a total of $N_e/2$ marks. We would have to make marks in the negative direction too (electrons moving to the left), so the largest momentum will be $(N_e/4)\Delta m_e v$. Now suppose the electrons are confined to a two-dimensional square sheet of paper. Show that, leaving out factors of order unity, their maximum momentum is $N_e^{1/2}\Delta m_e v$. (Hint: each pair of electrons occupies a *square* of momentum uncertainty.) Similarly, show for three dimensions that the result is $N_e^{1/3}\Delta m_e v$.

Solution 12.4.1

Suppose that, in the two-dimensional case, the maximum momentum in either direction is called p_2. Consider a plane with coordinates equal to the momentum in the x-direction and the momentum in the y-direction. In this plane, draw a circle of radius p_2. All particles must have momentum vectors within this circle. Since only at most two particles can have the same momentum to within an uncertainty Δp, we can divide the circle up into squares of size Δp. Since each square has area $(\Delta p)^2$, there are $\pi(p_2/\Delta p)^2$ such squares (neglecting the problem that squares out at the boundary of the circle don't pack properly, which is a small effect if the number of squares is large). Since there are two particles per square, the total number of particles is $N = 2\pi(p_2/\Delta p)^2$. Solving this for p_2 gives $p_2 = \Delta p(N/2\pi)^{1/2}$. This verifies the desired result, since we are neglecting factors of order unity (in this case, the factor of 2π).

For the three-dimensional case, the maximum momentum p_3 gives the size of a sphere in three momentum dimensions. The calculation goes through in a similar way.

Exercise 12.5.2: *Relativistic degenerate gas equation of state* [page 146]

Find the constant β in Equation 12.19 on page 146.

Solution 12.5.2

Equation 12.16 on page 146 needs to be modified to involve the mass density ρ. The mass density is dominated by the protons and neutrons, and there are μ of them per electron. Therefore the density is $\rho = \mu m_p N_e/V$. Solving this for N_e, we get $N_e = V\rho/\mu m_p$. Putting this into Equation 12.16 on page 146, we get

$$p = \frac{hcV^{1/3}}{3R\mu^{4/3}m_p^{4/3}}\rho^{4/3}.$$

This is of the form of Equation 12.19 on page 146. To get the coefficient β, we need to simplify the expression. Replacing V by $4\pi R^3/3$, we find that the size R of the star drops out and we just get

$$\beta = \frac{4^{1/3}\pi^{1/3}hc}{3^{4/3}\mu^{4/3}m_p^{4/3}}.$$

Gravity from the ground up

Solutions to selected exercises

Chapter 13

Exercise 13.4.1: ***Accretion disk temperatures*** [page 161]

Perform the arithmetic to get the temperature $T = 8 \times 10^4$ K in Equation 13.11 on page 161. Then use this equation to calculate the other values of temperature given in the text.

Solution 13.4.1

The starting point is Equation 13.10 on page 161, $T = (GMM_t/\pi\sigma R^3)^{1/4}$. The given values, converted to SI units, are

$$M = 1M_\odot = 2 \times 10^{30}\,\mathrm{kg},$$

$$M_t = 10^{-10}M_\odot\,\mathrm{y}^{-1} = 2 \times 10^{20}\,\mathrm{kg\,y}^{-1} = 6.3 \times 10^{12}\,\mathrm{kg\,s}^{-1},$$

and $R = 5 \times 10^6$ m. The fundamental constants are $G = 6.67 \times 10^{-11}$ and $\sigma = 5.67 \times 10^{-8}$ in SI units. Putting all these into the formula for temperature gives $T = 7.8 \times 10^4$ K.

To get the temperature of a disk around a neutron star, we just need to change the radius from 5×10^6 m to 10^4 m. The ratio of these radii is 500, and the temperature depends on the radius to the power $3/4$. This leads to an increase of temperature by a factor of $500^{3/4} = 106$. The temperature of this disk is then 8.2×10^6 K.

Exercise 13.4.2: ***Accretion disk luminosities*** [page 161]

Take the equation $L = GMM_t/R$ for the disk luminosity and write it in a similar normalized form to Equation 13.11 on page 161, scaling the mass of the central object to $10^9M_\odot$, the accretion rate M_t to $1M_\odot\,\mathrm{y}^{-1}$, and the radius to 10^{13} m. These values are appropriate to accretion disks around the giant black holes that power quasars (Chapter 14).

Solution 13.4.2

Here we want to evaluate the luminosity for

$$M = 10^9M_\odot = 2 \times 10^{39}\,\mathrm{kg},$$

$$M_t = 1M_\odot\,\mathrm{y}^{-1} = 2 \times 10^{30}\,\mathrm{kg\,y}^{-1} = 6.3 \times 10^{22}\,\mathrm{kg\,s}^{-1},$$

and $R = 10^{13}$ m. Putting this into the luminosity formula, we get $L = 8 \times 10^{38}$ W. This is over 2×10^{12} times the luminosity of the Sun, which means it is comparable to the luminosity of an entire large galaxy!

Expressed as a scaling formula, we find

$$L = 8 \times 10^{38}\left(\frac{M}{10^9M_\odot}\right)\left(\frac{M_t}{1M_\odot\,\mathrm{y}^{-1}}\right)\left(\frac{R}{10^{13}\,\mathrm{m}}\right)^{-1}.$$

Gravity from the ground up

Solutions to selected exercises

Chapter 14

Exercise 14.1.1: *Binding energy of a cluster* [page 165]
Show that $E_{\rm boil}$ can be expressed as

$$E_{\rm boil} = \frac{GM_{\rm cl}^2}{R_{\rm cl}}. \tag{14.5}$$

In this form it is usually called the *binding energy* of the cluster. This is only an approximation, of course, accurate to a factor of two or so.

Solution 14.1.1

To boil off the cluster one must give all the stars a velocity equal to $v_{\rm escape}$. This means adding a kinetic energy of $\frac{1}{2}M_{\rm cl}v_{\rm escape}^2$. Using $v_{\rm escape} = (2GM_{\rm cl}/R_{\rm cl})^{1/2}$, we get the required result:

$$E_{\rm boil} = \frac{1}{2}M_{\rm cl}\frac{2GM_{\rm cl}}{R_{\rm cl}} = \frac{GM_{\rm cl}^2}{R_{\rm cl}}.$$

Gravity from the ground up
Solutions to selected exercises
Chapter 15

Exercise 15.1.1: ***More photon velocities*** [page 183]

Let $v = -c$ in Equation 15.3 on page 183, corresponding to a photon moving backwards relative to the one we tested above. Show that again $V = -c$: the speed of the photon does not depend on the observer.

Solution 15.1.1

The numerator in Equation 15.3 on page 183 becomes $u - c$ and the denominator becomes $1 - u/c = -(u - c)/c$. The fraction therefore evaluates to $-c$, which is the same speed as the input speed of the photon. Thus, this composition law respects the speed of light in both directions.

Exercise 15.1.2: ***Computing the graph*** [page 183]

In Figure 15.2 on page 184 we plot the composition law Equation 15.3 on page 183 for the special case $u = 0.4c$. Compute V/c for the set of values $v = \{0.1c, 0.4c, 0.9c\}$. Compare them with the points plotted on the curve in the right-hand panel of the figure.

Solution 15.1.2

The values given by Equation 15.3 on page 183 for $u = 0.4c$ and the given values of v are

$$v = 0.1c \qquad V = (0.4c + 0.1c)/(1 + 0.4 \cdot 0.1) = 0.48c;$$

$$v = 0.4c \qquad V = (0.4c + 0.4c)/(1 + 0.4 \cdot 0.4) = 0.69c;$$

$$v = 0.9c \qquad V = (0.4c + 0.9c)/(1 + 0.4 \cdot 0.9) = 0.96c;$$

These lie on the solid curve in the right-hand panel of Figure 15.2 on page 184.

Exercise 15.1.3: ***How fast is relativistic?*** [page 183]

If both u and v are $0.1c$, what is the fractional error in using the Galilean addition law? [The fractional error is the difference between the Einstein and Galilean results (the error), divided by the Einstein result (the correct answer).] If $u = v = 0.3$, what is the fractional error? Suppose V can be measured to an accuracy of $\pm 5\%$. What is the largest speed (again assuming $u = v$) for which one can use the Galilean formula and make errors too small to be measured?

Solution 15.1.3

The relativistic law for speeds of $0.1c$ and $0.1c$ gives a result

$$V = (0.1c + 0.1c)/(1 + 0.1 \cdot 0.1) = 0.198c.$$

If we were to use the Galilean law, we would get exactly $0.2c$. The difference between the results is $0.002c$, which gives a fractional error of $0.002c/0.198c = 0.01$. That is, the Galilean law is in error by 1% in this case.

Doing the same for two speeds of $0.3c$ gives a larger fractional error. First, the Einstein composition law gives

$$V = (0.3c + 0.3c)/(1 + 0.3 \cdot 0.3) = 0.55c.$$

The Galilean law would give simply $0.6c$, leading to an error of $0.05c$. As fraction of the correct answer, this error is $0.05c/0.55c = 0.09$, or 9%.

If our measurement accuracy for speeds is ±5%, we won't notice the error if the speeds are somewhere between our two examples. Trial and error leads to $u = v = 0.22c$, for which the Einstein law gives $V = 0.42c$ while the Galilean gives $0.44c$. The error is $0.02c$ and the relative error is 0.05 or 5%. So if one is composing identical speeds smaller than $0.22c$, one can use the Galilean law to an accuracy of better than 5%.

Exercise 15.1.4: *Zero is still zero* [page 183]

The Einstein composition law still has some features that we expect from everyday life (and logical consistency). Show that, if the projectile remains at rest with respect to the moving experimenter (so $v = 0$), then its speed relative to the experimenter at rest is the same as the speed of the moving experimenter, $V = u$. Show further that if the moving experimenter shoots the projectile backwards with a speed of $v = -u$, then it will be at rest with respect to the resting experimenter ($V = 0$).

Solution 15.1.4

If $v = 0$, then $V = (u + 0)/(1 + 0) = u$. This is required by logical consistency, since all objects that are at rest with respect to one another must be measured by any other experimenter to have the same speed as one another.

Similarly, if $v = -u$ then $V = (u - u)/(1 - u^2/c^2) = 0$. This is again required by logical consistency. If the projectile with speed v is thrown backwards relative to the moving object with the speed $-u$, then it will have the same speed relative to the moving object as the experimenter who is at rest, and therefore it must be at rest relative to this experimenter as well. The result of 0 verifies that the Einstein law respects this.

Exercise 15.3.1: *Slow-velocity expansion* [page 192]

Use the binomial expansion Equation 5.1 on page 43 to show that the expansion of $(1 - v^2/c^2)^{1/2}$ for small v/c is

$$\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots.$$

Solution 15.3.1

The binomial theorem as given in Equation 5.1 on page 43 is

$$(a + b)^n = a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots.$$

We want to expand $(1 - v^2/c^2)^{1/2}$, so we take $a = 1$, $b = -v^2/c^2$, and $n = 1/2$. Then the formula above becomes (using only the first two terms)

$$\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2} = 1 + \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \cdots.$$

This simplifies to the required expression

$$\left[1 - \left(\frac{v}{c}\right)^2\right]^{1/2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + \cdots.$$

Notice that the $\cdots$ represents terms that are *smaller* that the terms written out explicitly in this expression, provided that $|v/c| < 1$, which is always the case. The reason they are smaller is that they involve higher powers of v/c, and each time v/c is multiplied by itself the result gets smaller. So the term we have kept is the largest part of the correction introduced by relativity.

Exercise 15.3.2: *How much mass is in kinetic energy?* [page 192]

Consider the example given in the text, of an automobile with a rest-mass of 1000 kg. Show that its kinetic energy at a speed of $100\,\mathrm{km\,hr^{-1}}$ has a mass equivalent of 4 ng (4 nanograms, or 4×10^{-12} kg).

Solution 15.3.2

Since the speed is very small compared to c, we just use the non-relativistic expression for the kinetic energy $\frac{1}{2}mv^2 = 3.9 \times 10^5$ J. (Remember to convert the speed of $100\,\mathrm{km\,hr^{-1}}$ to its SI value $28\,\mathrm{m\,s^{-1}}$.) To find the mass equivalent, divide this energy by c^2, which gives 4.3×10^{-12} kg, or 4.3×10^{-9} g, or 4.3 nanograms.

Gravity from the ground up

Solutions to selected exercises

Chapter 18

Exercise 18.1.1: *Redshift near the Sun* [page 231]

Derive Equation 18.8 on page 231, starting from Investigation 2.2 on page 16. Calculate the redshift experienced by a photon with a wavelength of $0.5\,\mu$m as it travels from the surface of the Sun to a very distant observer. Calculate the redshift of the same photon if it is observed by a space observatory in the Earth's orbit but far from the Earth. Finally, calculate the redshift if the same photon is observed by an astronomer on the surface of the Earth.

Solution 18.1.1

The first part of this exercise, proving the redshift formula, is difficult. The remaining parts are simply putting numbers into the formula.

The thing that makes the derivation of the redshift in Investigation 2.2 on page 16 relatively simple is the fact that the distance h is small compared to the radius of the Earth, so that the acceleration of gravity g can be taken to be constant during the time that the experiment is performed. If we now want to do a redshift experiment between an experimenter at a distance r from a body of mass M and another experimenter very far away, we cannot simply use the formula gh/c^2: g does not have a single value over this distance and h, the separation of the two experimenters, is very large.

Instead, we have to do the calculation in short steps, small enough to approximate g as constant over the small distance. Thus, let us imaging a series of radii extending outward from r in steps of size Δr: $r_0 = r$, $r_1 = r + \Delta r$, $r_2 = r + 2\Delta r$, $r_3 = r + 3\Delta r$, and so on out to whereever the distant experimenter is located. In each of these small steps, the redshift depends on the local value of g, which is of course $g_n = GM/{r_n}^2$ at the location r_n. The redshift factor in step at r_n will be $g_n\Delta r/c^2 = (GM/r_nc^2)(\Delta r/r_n)$. I have grouped these factors in this way to show that the redshift at any step is small: while the factor GM/r_nc^2 can be of order unity in a very relativistic situation, it is small for Newtonian gravity; and the factor $\Delta r/r_n$ is as small as we wish to make it, since we can control how large Δr is.

Now, the total redshift will be the product of all the redshift factors

$$(1 + g_0\Delta r/c^2) \cdot (1 + g_1\Delta r/c^2) \cdot (1 + g_2\Delta r/c^2) \cdot \cdots .$$

This is not as complicated as it looks. When two factors are multiplied together, one of terms in the result is so small that it can be neglected. Thus, let us do the product

$$(1 + g_0\Delta r/c^2) \cdot (1 + g_1\Delta r/c^2) = 1 + g_0\Delta r/c^2 + g_1\Delta r/c^2 + g_0\Delta r/c^2 \cdot g_1\Delta r/c^2 .$$

The final term can be neglected, since it depnds on $(\Delta r/r)^2$, which will be very small. Thus, we will take as a good approximation the simpler formula

$$1 + g_0\Delta r/c^2 + g_1\Delta r/c^2 + g_2\Delta r/c^2 + \cdots .$$

Using our earlier expression for the individual redshift factors, we get

$$1 + \frac{GM\Delta r}{c^2}\left(\frac{1}{r_0^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \cdots\right).$$

The problem is to show that this gives the total redshift shown in Equation 18.8 on page 231. Those readers who understand the integral calculus can show this directly. We will choose a different route, namely to show that if we start with Equation 18.8 on page 231 and take small steps of size Δr, then the differences in redshift from one step to another are the same as given above. To be concrete, let us ask what is the *difference* in the redshift between a location r_n and r_{n+1}. This is

$$\begin{aligned}\left(1 - \frac{GM}{r_{n+1}c^2}\right) - \left(1 - \frac{GM}{r_n c^2}\right) &= \frac{GM}{c^2}\left(-\frac{1}{r_n + \Delta r} + \frac{1}{r_n}\right)\\ &= \frac{GM}{c^2}\left[-(r_n + \Delta r)^{-1} + r_n^{-1}\right]\\ &= \frac{GM}{c^2}\left(-r_n^{-1} + r_n^{-2}\Delta r + \cdots + r_n^{-1}\right),\end{aligned}$$

where in the last line we have applied the binomial theorem, Equation 5.1 on page 43, and kept only the first two terms since the successive terms are smaller and smaller (containing higher powers of $\Delta r/r$). Simplifying gives

$$\frac{GM}{c^2}\frac{\Delta r}{r_n^2}.$$

This is identical to the contribution of r_n to the full redshift factor, as calculated at the end of the previous paragraph. This shows that the expression given in Equation 18.8 on page 231 is the correct expression for large changes in r.

To do the rest of the problem, note that the change in wavelength is found by multiplying the factor GM/rc^2 by the original wavelength. For the Sun's mass and radius this factor is 2.12×10^{-6}. If the original wavelength was $0.5\,\mu\text{m} = 5 \times 10^{-7}\,\text{m}$, then the change in its wavelength was a mere $1.061 \times 10^{-12}\,\text{m}$, or 1 pm. The sense of the change in wavelength is that it got longer.

If the observer is not very far from the Sun, but only in the orbit of the Earth, then the redshift is given by the *difference* of the redshift factors at the two radii. The Earth's orbit is at $1\,\text{AU} = 1.5 \times 10^{11}\,\text{m}$, so the redshift factor there is $GM_\odot/(1\,\text{AU})c^2 = 10^{-8}$. The difference from the redshift factor at the Sun's surface is 2.11×10^{-6}, leading to a redshift of $1.055 \times 10^{-12}\,\text{m}$. So an observer at the Earth's orbit would have to be able to measure the redshift to better than 1% accuracy to tell the difference from the redshift seen by a much more distant observer: the Earth is already "far away".

If the observer is sitting on the surface of the Earth, then the redshift is composed of two factors: the redshift experienced by the photon reaching the Earth's orbit, calculated in the last paragraph, and then the *blueshift* experienced by the photon as it falls onto the surface of the Earth. This is a blueshift because the photon is moving more deeply into the Earth's gravitational field, rather than moving away. To calculate this blueshift factor, we need to evaluate GM/rc^2 using the Earth's mass and radius; this gives 7×10^{-10}. Because this is a blueshift, the wavelength of the photon *shortens* by $3.5 \times 10^{-16}\,\text{m}$, or 0.35 fm. The net redshift is therefore $1.0547 \times 10^{-12}\,\text{m}$.

Gravity from the ground up

Solutions to selected exercises

Chapter 19

Exercise 19.2.1: ***Upper bound on the cosmological constant*** [page 257]

The fact that Newtonian gravity describes the orbits of planets in the Solar System very well, using only one parameter (the mass of the Sun) for all planetary orbits, suggests that the cosmological constant must create a smaller mass density than the mean mass of the Solar System out to Pluto's orbit. (a) Calculate this mean density by dividing the mass of the Sun by the volume of a sphere whose radius is the radius of Pluto's orbit. (b) From this, calculate the value of the cosmological constant Λ that would give a mass density ρ_Λ of the same value. Use Equation 19.7 on page 255.

Solution 19.2.1

(a) The mean density of the Sun out to Pluto's orbit is the mass of the Sun, $M_\odot$, divided by the volume of a sphere whose radius is Pluto's orbital radius, 6×10^{12} m. This works out to be $2.2 \times 10^{-9}\,\mathrm{kg\,m^{-3}}$.

(b) The value of the cosmological constant that would have this mass density can be obtained from the density by multiplying by $8\pi G$, as one can deduce from Equation 19.7 on page 255. This gives $4 \times 10^{-18}\,\mathrm{s^{-2}}$.

Gravity from the ground up

Solutions to selected exercises

Chapter 20

Exercise 20.1.1: ***How big is the nuclear hard core?*** [page 265]

Use the mass 1.67×10^{-27} kg of a nucleon and the density $2 \times 10^{17}\,\mathrm{kg\,m^{-3}}$ to calculate the volume occupied by each nucleon in a nucleus. If the nuclei are contained in cubical boxes, how big is each box? What is the size of the hard core, the irreducible radius of a nucleon?

Solution 20.1.1

The density of a nucleus is its mass divided by its volume. If we consider a single nucleon, then it must occupy an average volume whose size is just what is needed to give a density equal to nuclear density. Thus, we find its mean volume by dividing its mass by the nuclear density: $V = m_{\mathrm{p}}/\rho_{\mathrm{nuclear}} = 8 \times 10^{-45}\,\mathrm{m}^3$. If this volume is a cube, then it has a side which is the cube-root of this number, or $2 \times 10^{-15}\,\mathrm{m} = 2\,\mathrm{fm}$. If the nucleon is a sphere sitting in the middle of this box, then its radius would be half the size of the box, or 1 fm.

Exercise 20.1.2: ***Calculating the minimum neutron star mass*** [page 265]

Solve Equation 20.1 on page 265 for M and use the value of ρ_{nucl} in the previous exercise to verify the minimum mass in Equation 20.2 on page 265.

Solution 20.1.2

Solving Equation 20.1 on page 265 for M gives

$$M = 0.17 G^{-3/2} v_{\mathrm{escape}}^3 \rho_{\mathrm{nucleus}}^{-1/2}.$$

If we put the nuclear escape speed $v_{\mathrm{escape}} = 4 \times 10^7\,\mathrm{m\,s^{-1}}$ into this expression and use the nuclear density $\rho_{\mathrm{nucl}} = 2 \times 10^{17}\,\mathrm{kg\,m^{-3}}$, we get $M = 4 \times 10^{28}$ kg, as required.

Exercise 20.1.3: ***What does a neutron star look like?*** [page 265]

Taking the mass of a neutron star to be $1M_\odot$, what is its radius? What is the escape speed of a projectile leaving its surface? What is the speed with which a projectile falling from rest far away reaches the surface? What fraction of the rest -mass of such a projectile is its kinetic energy when it arrives at the surface? What is the orbital speed of a particle in a circular orbit just above the surface of the star? What is its orbital period? Do all calculations using Newtonian gravity, even though the speeds are relativistic.

Solution 20.1.3

Given a density equal to the nuclear density $\rho_{\mathrm{nucl}} = 2 \times 10^{17}\,\mathrm{kg\,m^{-3}}$, a star with the mass of the Sun would have a volume equal to $1M_\odot/\rho_{\mathrm{nucl}} = 10^{13}\,\mathrm{m}^3$. The radius of a sphere of this volume is $R = 13\,\mathrm{km}$.

The escape speed of a projectile leaving its surface is $v_{\mathrm{esc}} = (2GM_\odot/R)^{1/2} = 1.4 \times 10^8\,\mathrm{m\,s^{-1}} = 0.48c$. This is the same speed that a projectile falling from rest far away would have when it reached the surface.

The orbital speed near the surface is $v_{\rm orb} = (GM_\odot/R)^{1/2} = 1.0 \times 10^8\,{\rm m\,s^{-1}} = c/3$. The period is the time it takes to travel the circumference of a circle of radius R at this speed, which is $P = 2\pi R/v_{\rm orb} = 8 \times 10^{-4}\,{\rm s} = 0.8\,{\rm ms}$.

Note that we have used Newtonian formulas here, even though the speeds approach the speed of light. The results can only be approximate; we must expect that fully relativistic calculations will give answers that may differ by factors of order 2.

Exercise 20.1.4: *Thermal effects in neutron stars* [page 265]

If the binding energy of a nucleon is 8 MeV, what temperature would the star have to have in order to boil off a nucleon? Since the pressure support for the star comes from the hard-core repulsion and not from random thermal motions of the star, it is possible for stars to cool off after formation without changing their properties. Give an argument that a star is "cold" (thermal effects are unimportant for its structure) if its temperature is smaller than the one you have just calculated. Assume the star has a temperature of 10^6K. What is its black-body luminosity? (See Equation 10.3 on page 116.) What is the wavelength at which it is brightest? (See Equation 10.9 on page 117.)

Solution 20.1.4

For a neutron to "boil" off, its thermal kinetic energy $^3\!/_2 kT$ must equal the binding energy per nucleon of $8\,{\rm MeV} = 1.3 \times 10^{-12}\,{\rm J}$. This gives $T = 6 \times 10^{10}\,{\rm K}$.

If the true temperature of the star is smaller than this, then most of its pressure must be coming from non-thermal effects (the exclusion principle and the nuclear hard-core potential) rather than from thermal motions. In this sense the star is "cold".

If the neutron star's temperature is 10^6 K, then with the radius $R = 1.3 \times 10^4$ m calculated in the previous exercise we find a black-body luminosity of $L = 4\pi R^2 \sigma T^4 = 1.2 \times 10^{26}\,{\rm W}$. This is only about one-third the luminosity of the Sun, despite its much higher temperature. The wavelength at which it is brightest is 2.9×10^{-7} cm. This is in the high-ultraviolet or low-energy X-ray part of the electromagnetic spectrum.

Exercise 20.2.1: *Pulsar energy storehouse* [page 275]

A pulsar stores its energy as rotation. Estimate how much energy was released when the neutron star was formed by calculating the approximate gravitational potential energy of the neutron star, $-GM^2/2R$. You should find that the rotational energy is a small fraction of what was available when the star formed. What happened to the rest of the energy?

Solution 20.2.1

The energy released is $GM^2/2R = 10^{46}\,{\rm J}$. Given the Crab pulsar's kinetic energy of rotation, $1.5 \times 10^{42}\,{\rm J}$, we see that only about 0.01% of the available energy went into making the neutron star spin. The remainder had to be carried away from the star when it was formed. Most of this was probably taken by the emitted neutrinos, but an unknown fraction was carried away by gravitational radiation.

Gravity from the ground up

Solutions to selected exercises

Chapter 21

Exercise 21.3.1: ***Accretion disks*** [page 300]

(a) If the spectrum of an X-ray source looks like a black-body spectrum that peaks around 1 keV, show that the associated temperature of the body should be near 10^7 K. (b) If the luminosity of the X-ray source is 10^{30} W, estimate the surface area and effective radius of the region emitting the X-rays. (c) Find the rate at which mass accretes onto the compact object, assuming that its mass is $15M_\odot$. Express the result in units of solar masses per year.

Solution 21.3.1

(a) The temperature T is given approximately by setting kt equal to the energy of a typical photon. The photon energy 1 keV converts to 1.6×10^{-16} J. Dividing by k gives 1.2×10^7 K. This is a little low for CYG X-1, whose spectrum peaks around 2 keV, and whose temperature is therefore around 2.4×10^7 K. But CYG X-1 regularly switches between this low-temperature state and a high-temperature state in which the spectrum keeps rising toward 100 keV; in this state, however, the spectrum does not seem to be a black-body spectrum, which means that the accretion disk must be too thin to be a perfect absorber. So these X-ray sources can be much more complex than our simple model!

(b) Here we use the black-body luminosity formula $L = \sigma A T^4$, where A is the area of the black-body. This will be correct to some approximation. It should be used to get an "effective" area, an idea of how big the main emitting region of the disk is. It is not going to give us the total radius of the accretion disk, since the outer parts are cool and are not emitting X-rays. Using the temperature of 10^7 K and the given luminosity allows us to solve for the area, which comes out to be $1.8 \times 10^9\,\mathrm{m}^2$. Since this is a disk, this should equal πr^2. Solving for r gives $r = 24$ km.

This is a little smaller than the 30 km radius of a black hole of mass $15M_\odot$, which is what CYG X-1 is thought to be. So our assumption of a filled disk is not a good one. Although the problem did not ask us to do this, let us instead see what we should expect if the emission comes from a ring (annulus) near the hole. The inner edge of this ring should be at th e innermost stable circular orbit, which was discussed earlier in Chapter 21. Inside this orbit, no circular motion is possible, so material falls quickly into the hole. Any radiating material must accumulate outside this radius. It lies at three times the Schwarzschild radius, or about 133 km for the given mass. If we denote this radius by r then the area of a narrow ring of width Δr is $2\pi r \Delta r$. Using the earlier value for the area and solving for Δr, we find that the thickness of the annulus is $\Delta r = 2.1$ km, less than 2% of the radius. The black-body emission therefore seems to come from a very thin annulus near the innermost stable circular orbit.

(c) We use Equation 13.8 on page 161 with the given values of $M = 15M_\odot$ and $L = 10^{30}$ W. We take the radius R to be that of the innermost stable circular orbit of a black hole of this mass, for the reason explained above. This radius is $6GM/c^2 = 133$ km. Using these numbers, we find $M_t = 6.7 \times 10^{13}\,\mathrm{kg\,s^{-1}} = 1.1 \times 10^{-9} M_\odot\,\mathrm{y}^{-1}$.

Exercise 21.4.1: *Hawking radiation* [page 305]

Perform the computations indicated in this investigation. Then find out how much time the hole has remaining when its temperature is high enough to produce electrons in its radiation (this will require kT to exceed $m_e c^2$).

Solution 21.4.1

The computations in this investigation begin with the luminosity formula for black-bodies, and substitute into that the expression for the area of a black-hole, the temperature of hte black hole (Equation 21.13 on page 305), and the expression for the Stefan-Boltzmann constant (Equation 10.4 on page 116):

$$\begin{aligned} L &= \sigma A T^4 \\ &= \frac{2\pi^5 k^4}{15c^2h^3}\frac{16\pi G^2M^2}{c^4}\left(\frac{hc^3}{16\pi^2 kGM}\right)^4 \\ &= \frac{1}{30720\pi^2}\frac{c^6 h}{G^2M^2}. \end{aligned}$$

The expression in the investigation is obtained from this by grouping factors together in a convenient way.

Now, the remaining lifetime τ of a hole of mass M can be found approximately by assuming that its luminosity does not change with time, so that the energy radiated in a given time t is Lt. The hole will have evaporated when this equals its total mass-energy, Mc^2. This is only an approximation, but since the luminosity increases very rapidly as the mass of the hole decreases, the estimate will not be far off.

The lifetime is therefore found from the equation $L\tau = Mc^2$. Solving for τ gives

$$\tau = \frac{Mc^2}{L} = 30720\pi^2\left(\frac{GM}{c^3}\right)^3\frac{c^5}{hG}.$$

The expression contains a dimensionless constant times the cube of the light-crossing time across the black hole GM/c^3 divided by the square of the Planck time $t_{Pl} = (hG/c^5)^{1/2}$, defined in Equation 21.11 on page 295.

This is the remaining lifetime of any black hole. When the black hole is hot enough to emit electrons, its temperature equals $m_e c^2/k$, and its mass is (by Equation 21.13 on page 305)

$$M = \frac{hc}{16\pi^2 G m_e} = 2\times 10^{13}\,\text{kg} = 10^{-17}M_\odot.$$

Putting this into the expression above for τ gives

$$\tau = \frac{15}{2\pi^2}\left(\frac{h}{m_e c^2}\right)^3\frac{c^5}{hG}.$$

This evaluates to 2.2×10^{25} s $= 7\times 10^{17}$ y, or about 5×10^{7} times the age of the Universe!

Gravity from the ground up

Solutions to selected exercises

Chapter 22

Exercise 22.1.1: *Dimensional analysis* [page 315]

Fill in the missing steps above that show that the dimensions of energy flux are $\mathrm{kg\,s^{-3}}$. Then show similarly that the dimensions of c^3/G times the square of the frequency are the same.

Solution 22.1.1

Flux is energy per unit area per unit time. The dimensions of energy are $\mathrm{J} = \mathrm{kg\,m^2\,s^{-2}}$ (think of kinetic energy, mass times the square of velocity). Dividing by $\mathrm{m^2}$ to get energy per unit area gives $\mathrm{kg\,s^{-2}}$. Dividing by s to get energy per area per unit time yields, finally, $\mathrm{kg\,s^{-3}}$, as required.

The dimensions of c^3 are $\mathrm{m^3\,s^{-3}}$. The dimensions of G are $\mathrm{kg^{-1}\,m^3\,s^{-2}}$, so when we divide by this we get $\mathrm{kg\,s^{-1}}$. When multiplied by f^2, which has units $\mathrm{Hz^2} = \mathrm{s^{-2}}$ we get the dimensions of flux as required.

Exercise 22.1.2: *Size of gravitational wave flux* [page 315]

We saw that a gravitational wave arriving at the Earth might have an amplitude h as large as 3×10^{-21}. If its frequency is 1000 Hz, then calculate the energy flux from such a wave. Compare this with the flux of energy in the light reaching us from a full Moon, $1.5 \times 10^{-3}\,\mathrm{W\,m^{-2}}$. Use Equation 9.2 on page 108 to compute the apparent magnitude of the source. Naturally, the source is not visible in light, so this magnitude does not mean a telescope could see it, but it gives an idea of how much energy is transported by the wave, compared to the energy we receive from other astronomical objects.

Solution 22.1.2

Using Equation 22.4 on page 316 with the given values yields:

$$F = (\pi/4)(3 \times 10^8\,\mathrm{m\,s^{-1}})^3(6.67 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}})^{-1}(1000\,\mathrm{s^{-1}})^2(3 \times 10^{-21})^2 = 2.9\,\mathrm{kg\,s^{-3}}.$$

This is about 2000 times larger than the flux from the full Moon! Using Equation 9.2 on page 108 with this flux gives an apparent magnitude of -20. This is brighter than any star.

Exercise 22.3.1: *Radiation from example binaries* [page 321]

Do the calculations that lead to the values in Table 22.1 on page 319 for the orbital numbers and chirp times from the values of M, R, and r given in the table.

Solution 22.3.1

The formulas we need to use are Equation 22.6 on page 320 for the frequency in the fourth column, Equation 22.12 on page 321 for the fifth column, Equation 22.7 on page 320 for the sixth, and Equation 22.10 on page 320 for the final column. Remember to convert the radii given in kilometers into meters, and the distances given in kiloparsecs to meters. The calculation for the first line goes as follows: $f = 6.86 \times 10^{-5}\,\mathrm{s}^{-1}$; $t_{\mathrm{gw}} = 7.48 \times 10^{9}\,\mathrm{yr}$ (remember to convert the answer from seconds to years); $h = 3.46 \times 10^{-23}$; and $L_{\mathrm{gw}} = 1.43 \times 10^{-3} L_{\odot}$ (get the luminosity of the Sun $L_{\odot}$ from the Appendix).

The other lines of the table are similar. In fact, they can be found most easily by scaling; this means, for example, that the frequency in the second line should be higher by the $3/2$-power of the ratio of the orbital radii, which is a factor of 2.83×10^{6}. Multiplying this by the frequency in the first line, we get $194\,\mathrm{s}^{-1}$. This scaling method avoids having to use the values of c and G each time, recalculating only what is necessary.

Exercise 22.3.2: *Chirp times* [page 321]

From the chirp time for the system that resembles the Hulse–Taylor pulsar that was calculated in Exercise 22.3.1 on page 321, work out the rate of change of the period: what fraction of a second does the orbital period lose each second? Compare this with the measured number quoted in the text. Explain the difference. (See the caption for Table 22.1 on page 319.)

Solution 22.3.2

The rate of change of the period is the period divided by the chirp time (converted back into seconds). From the table the ratio of these two is 6.21×10^{-14}, which is dimensionless (the ratio of two times). In the text we quote the measured value for the Hulse-Taylor pulsar, 2.44×10^{-12}. This is a factor of 39.3 times large than our computed value. The main difference here is that we have assumed that the orbit is circular, whereas the real system has a highly eccentric orbit. Since the luminosity of the system is a strong function of the separation of the stars, the rate of shrinkage of an eccentric binary is much larger than that of a similar system in a cirular orbit with the same period.

Gravity from the ground up

Solutions to selected exercises

Chapter 23

Exercise 23.1.1: *Einstein ring* [page 337]

Perform the indicated algebra to derive the Einstein radius and its angular size.

Solution 23.1.1

The basic equation for the deflection angle, given in the investigation, can be worked into the following form by multiplying by b and putting the right-hand side over a common denominator:

$$\frac{4GM}{c^2} = b^2(D_{\rm LS} + D_{\rm L})/D_{\rm L}D_{\rm LS}.$$

Solving for b gives the required answer.

Gravity from the ground up

Solutions to selected exercises

Chapter 24

Exercise 24.1.1: ***Age of the Universe*** [page 348]

Assume that the Hubble constant has a value $70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. First convert this value to more standard units ($\mathrm{s^{-1}}$) by converting megaparsecs to kilometers. Then take its reciprocal to find that the approximate age of the Universe is 14 billion years. If the Hubble constant is larger, what does this do to the approximate age?

Solution 24.1.1

Converting kilometers to meters requires multiplying by *pwrteninline*3; converting megaparsecs to meters requires dividing by 3.09×10^{22}. The result is that the Hubble constant has the value $2.3 \times 10^{-18}\,\mathrm{s^{-1}}$. The reciprocal of this is a typical timescale for the expansion of the universe: $4.4 \times 10^{17}\,\mathrm{s}$, or $1.4 \times 10^{10}\,\mathrm{yr}$. Raising the Hubble constant makes the Universe expand faster and so reduces its apparent age.

Exercise 24.2.1: ***Energy density of the cosmic microwave background*** [page 356]

(a) Use Equation 24.7 on page 356 to calculate the energy density of the cosmic microwave background, given its temperature of $2.7\,\mathrm{K}$. (b) Show from this that the equivalent mass-density of the microwave background is $4.5 \times 10^{-31}\,\mathrm{kg\,m^{-3}}$.

Solution 24.2.1

(a) If we put in the values of the fundamental constants from the Appendix and use the given value of the temperature, we get $\epsilon_{\mathrm{bb}} = 4.02 \times 10^{-14}\,\mathrm{J\,m^{-3}}$.

(b) We get a mass density by converting energy to mass by dividing by c^2: $\rho_{\mathrm{bb}} = 4.5 \times 10^{-31}\,\mathrm{kg\,m^{-3}}$.

Exercise 24.2.2: ***Motion through the cosmic background*** [page 356]

According to measurements by COBE, the temperature of the cosmic microwave background has a maximum value on the sky that is $3.15\,\mathrm{mK}$ warmer than the average, and it has a minimum in a diametrically opposite direction that is $3.15\,\mathrm{mK}$ cooler than the average, after correcting for the motion of the satellite around the Earth and that of the Earth around the Sun. (The abbreviation mK stands for *millikelvin*.) Give an argument to show that the observed radiation should be black-body at a red- or blueshifted temperature. Then show that the speed of the Sun relative to the cosmic rest frame is $3.5 \times 10^{5}\,\mathrm{m\,s^{-1}}$.

Solution 24.2.2

A sufficient argument is that, since a black body is defined to be a body that absorbs light, a body should be a black body regardless of the speed of the observer: it should be a relativistically invariant notion. Therefore, if a body such as the Universe emits black body radiation as seen by one experimenters, it should emit black body radiation as seen by all other experimenters. The only thing that can change with the velocity of the experimenter is the observed temperature.

If we take the expression for the flux from a black body, Equation 10.1 on page 111, and replace the wavelength λ by $\lambda_z = (1+z)\lambda$, which is the effect of a redshift, then we see that if we define a new temperature $T_z = T/(1+z)$, we get

$$F_{\lambda(\text{redshifted})} = (1+z)^{-5}\frac{2\pi hc^2}{\lambda_z^5}\frac{1}{e^{hc/\lambda_z T_z}}.$$

This shows that the shape of the curve is exactly that of a black body with the new temperature, but the height of the curve — the total energy flux — is reduced by a large factor.

This reduction is not a contradiction to the fact that the flux emitted by a blackbody depends only on its temperature. Equation 10.1 on page 111 gives us the flux inside the black-body or just at its surface. Clearly, the flux must fall off with distance if we are outside the black body. Moreover, if the black body is moving away from us then there will be other redshift factors. The numerous factors of $1+z$ affecting the amplitude of the cosmic black body radiation come from the fact that we are observing radiation emitted by a surface that is very far away (because the radiation was emitted very long ago) and that is receding from us at a high speed (as a result of the cosmological expansion). What is not affected by distance or speed is the *spectrum*, the shape of the curve.

Now, the result of this is that we can interpret the observed spectrum as telling us that we are moving toward the source of the black body radiation in one direction and away from it in the opposite direction. This is consistent if the source is the Universe as a whole, or more properly if all parts of the Universe radiate the same radiation. The speed is just deducible from the redshift: $v = cz$. Since the change in the temperature is $\delta T = zT$, we have $v = c\delta T/T = 3.5 \times 10^5\,\mathrm{m\,s^{-1}}$.

Exercise 24.3.1: *The emptiness of the Universe* [page 363]

Since the luminous mass in galaxies is primarily in hydrogen, what would be the mean volume occupied by a single hydrogen atom if the mass in galaxies were smoothed out over the entire Universe? (Use the mean density of visible matter given in the text, $5 \times 10^{-29}\,\mathrm{kg\,m^{-3}}$.)

Solution 24.3.1

The mass of a hydrogen atom is about 1.67×10^{-27} kg, so the number of them per unit volume is the density divided by the mass: $2 \times 10^{-28}\,\mathrm{kg\,m^{-3}}/1.67 \times 10^{-27}\,\mathrm{kg} = 0.12$ atoms per cubic meter. Inverting this gives about 8 cubic meters per atom of hydrogen. Space is really very empty indeed!

Exercise 24.3.2: *Local accelerations* [page 363]

The nearest large galaxy to us is the Andromeda galaxy (also called M31), which is about 0.5 Mpc away and is falling towards our Galaxy, not receding from it. Take the mass of our Galaxy to be $10^{11} M_\odot$ and calculate the gravitational acceleration produced by our galaxy on the Andromeda galaxy, using the formula $a = -GM/r^2$. Calculate the cosmological acceleration given by Equation 24.17 on page 363 at a distance of 0.5 Mpc, using the critical density ρ_c. Compare the two accelerations. Are motions within the local group of galaxies (those dominated by Andromeda and ourselves) strongly affected by the expansion of the Universe?

Solution 24.3.2

The acceleration of M31 towards our Galaxy is GM/r^2, where $M = 10^{11} M_\odot = 2 \times 10^{41}$ kg and $r = 0.5\,\mathrm{Mpc} = 1.5 \times 10^{22}\,\mathrm{m}$. The arithmetic gives $a = 6 \times 10^{-14}\,\mathrm{m\,s^{-2}}$. The cosmological acceleration would be $2 \times 10^{-14}\,\mathrm{m\,s^{-2}}$. Now, since Andromeda is already falling towards us, the local group is gravitationally bound. The conclusion is that the local gravitational forces within the group will dominate any cosmological effects: regardless of what the Universe does, the local group will remail bound together. Cosmological effects will dominate only over larger distance scales.

Exercise 24.3.3: *Relation between Ω and q* [page 363]

Derive Equation 24.14 on page 362 from Equations 24.11, 24.10, and 24.18.

Solution 24.3.3

From Equation 24.10 on page 361 and Equation 24.11 on page 361 we find

$$\Omega = \frac{\rho}{3H_0^2/8\pi G} = \frac{8\pi G\rho}{3H_0^2}.$$

The result follows immediately if this is compared with Equation 24.18 on page 363.

Gravity from the ground up
Solutions to selected exercises
Chapter 25

Exercise 25.1.1: ***Radiation-dominated universe*** [page 371]

Find the dependence of the scale-factor on time for the radiation-dominated Universe. The analysis is similar to our derivation for the matter-dominated Universe above. The only difference is that, as explained above, the factor $\rho + 3p/c^2$ is proportional to R^{-4}. Show from this that $R \propto t^2$.

Solution 25.1.1

Following the same steps as in the investigation, we find that the acceleration equation imples that R/t^2 should be proportional to $1/R^3$, since the density is proportional to $1/R^4$ and this is multiplied by R (which is contained in d) to get the acceleration. Solving for R gives $R \propto t^2$.

Exercise 25.2.1: ***Random clumping*** [page 378]

Experiment with random clumping using a tossed coin as your random-number generator. Use three successive tosses to generate a number between 0 and 7, using its binary representation. That is, if the coin comes up heads assign a 1 to a digit, and if tails a 0. With three tosses you get three digits, say 010, and that is the number 2. (The digits abc represent the number $4a + 2b + c$.) Record each such number you get. Generate a large set of them, say 80. Each number should come up on average ten times, but some will come up more often and some less, at random. The excess over the average should be, according to the argument above, $10^{1/2} \approx 3$. You should expect some numbers to come up at least 13 times, and others only 7. You might expect one bin to have twice as large a fluctuation, i.e. to reach 16 or 4. Now go on and do twice as many, 160 numbers. (You need 480 coin tosses to do this!) Then the average will be 20 and the expected fluctuation $20^{1/2} \approx 4.5$. Although the fluctuation is larger in this case, it is a smaller fraction of the average, so that the distribution of numbers among the bins is actually smoother. If you have the stamina, go to 320 numbers. Verify that the typical fluctuation is of order six.

Solution 25.2.1

Here is a table of the results of my coin-tossing. It shows the number of times a given value of the three-digit binary number came up in each set of trials, and for comparison the number of times it would be expected to come up if all numbers were equally represented. It also shows the average of all the three-digit binary numbers that came out in each trial.

Trials	Expected	0's	1's	2's	3's	4's	5's	6's	7's	Average
80	10	9	8	10	12	13	9	10	9	3.55
160	20	24	19	26	17	16	21	20	17	3.32
320	40	42	39	36	37	39	47	40	40	3.54

Now, we see that the maximum excursion away from the expected value of 10 in the first set of trials is 3: there were 13 occurences of the number 4, while only 10 were expected. This is consistent with the idea that fluctuations of order $10^{1/2}$ are expected. In the second set of trials, with 160 numbers generated at random, the expected number of occurrences of each value would be 20, and the maximum deviation from this was 6: the number 2 came up 26 times. This is a little larger than the expected fluctuation of $20^{1/2}$, which is about 4.5. In the final trial, where I generated 320 random numbers, we expect each to occur 40 times and the maximum difference from this is 7: the number 5 came up 47 times. This is consistent with the expected fluctuation of $40^{1/2}$, or 6.3.

In random trials, anything can happen, but if there are enough trials then they should behave on average in a way consistent with expectations. Thus, in the trial which generated 160 numbers, the maximum fluctuation was slightly large, but if this trial were repeated many times there would also be trials in which the fluctuations were smaller than expected. If you want to explore this more effectively, don't use a coin: use a computer!