for example, would never form the kind of electromagnetic waves that radios and mobile phones use if no two of them could be the same. Lasers, which essentially emit strong beams of light with all the photons in exactly the same wavelength and state of oscillation, exist only because photons like being the same as one another. Particles that obey the exclusion principle are called fermions, after Fermi. Particles that like being together are called bosons, after the Indian physicist Satyendra Nath Bose (1854–1948).

Although degenerate matter has strange properties, the natural evolution of a star brings it to the point where degeneracy becomes important. Astronomers see white dwarfs in their telescopes, and they have just the size we have calculated. These huge objects depend for their very existence on the strange physics of quantum theory. Because they are composed of matter whose structure cannot be described in the conventional language of forces, they would have been incomprehensible to Newton. Yet they are abundant: one in ten stars is a white dwarf; and, as we noted in Chapter 10, one of the brightest stars in the night sky, Sirius, has a white dwarf in orbit about it.

The Chandrasekhar mass: white dwarfs can't get too heavy

Unfortunately for some stars, but fortunately for the evolution of life on Earth, the story of degeneracy does not stop here. The problem is that, if the star is very massive, then gravity will force the degenerate electrons into such a small volume that their typical speed becomes close to the speed of light. In this case, we have to treat the electrons by the rules of special relativity.

In this section: the kind of quantum-mechanical support that white dwarfs use can only support a little more than the mass of the Sun. With more mass, the star will collapse.

We will study special relativity beginning in Chapter 15, but here we need only one new fact that is explained there: the momentum carried by a photon is just proportional to its energy, in fact is E/c. This is very different from the situation for low-speed electrons, where the momentum is twice the kinetic energy divided by the speed of the particle. The difference is in part due to the fact that in special relativity, mass has energy, so the relationship between energy and momentum must include the total mass-energy of a particle. Now, since at a speed close enough to the speed of light, any particle behaves more and more like a photon, the momentum carried by a fast electron is also just its energy divided by c.

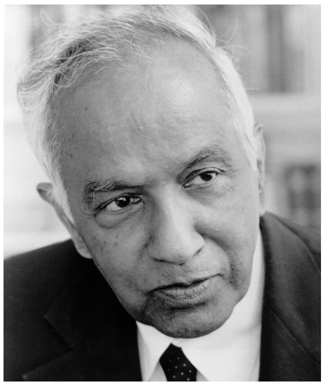

Figure 12.6. *Subrahmanyan Chandrasekhar made many important contributions to astrophysics, sharing the 1983 Nobel Prize for Physics. His discovery as a very young man of the limiting mass for white dwarfs led to a bitter conflict with Eddington (see Chapter 4), who felt that Nature simply could not behave in such a way! Chandrasekhar lost this battle, escaping from Cambridge to Chicago. Subsequent research, of course, vindicated his work completely. Chandrasekhar's modest manner, his devotion to science, and his erudition won him the respect and affection of generations of scientists. Image courtesy University of Chicago.*

Now, the energy of the gas particles is directly responsible for the pressure of the gas, so when the electrons become relativistic, the pressure starts to increase only in proportion to the uncertainty momentum, rather than to its square. This significantly weakens the degeneracy pressure that electrons can exert, and in fact we show in Investigation 12.5 on the next page that it leads to a universal *maximum mass of a white dwarf*, the maximum mass that can be supported by degenerate electrons. This mass is called the *Chandrasekhar mass* $M_{\rm Ch}$, after the Indian astrophysicist Subrahmanyan Chandrasekhar (1910–1995) who discovered it. Its value is in the range 1.2 to 1.4 times the mass of the Sun, depending on the exact composition of the star when it reaches the density of the white dwarf.

> The Chandrasekhar mass is one of the most remarkable numbers in all of physics. As we show in Investigation 12.5 on the following page, it depends mainly on some fundamental constants of nature, not on fine details of atomic or nuclear physics.

It seems to be an accident of our Universe that this particular combination of the constants of nature gives a mass for a relativistic degenerate white dwarf that is similar to the masses of ordinary stars. If this mass had come out to be much larger or much smaller than a solar mass, the death of most stars would be radically

Investigation 12.5. Deriving the Chandrasekhar Mass

For stars with relativistic electrons, we need to re-calculate the structure and equation of state from the beginning, since Equation 12.9 on page 143 is wrong for this case. As we will see in Chapter 15, the energy of a photon is just the speed of light times its momentum. Therefore, this must be almost true even for ordinary particles moving at close to the speed of light:

$$E = pc. \qquad (12.15)$$

Given the same uncertainty in momentum, $\Delta m_e v = h/2R$, using the exclusion principle to give a typical momentum that is a factor of $N_e^{1/3}$ larger than this, and then following the same steps as before for the ideal gas, we arrive at a pressure

$$p = \frac{hcN_e^{4/3}}{3RV}. \qquad (12.16)$$

Setting this equal to $3GM^2/4\pi R^4$ as in Investigation 12.3 on page 143, and replacing the volume by $4\pi R^3/3$, we get

$$\frac{hcN_e^{4/3}}{4\pi R^4} = \frac{3GM^2}{4\pi R^4}. \qquad (12.17)$$

Here we notice a remarkable and unexpected thing: the radius of the star drops out, and we are left with an equation that determines a single mass! This mass is the *unique* mass of a fully relativistic white dwarf. For non-relativistic white dwarfs we could choose a mass and find a radius, or vice versa. Here, we have no choice about the mass, and presumably the radius can be anything at all! This rather remarkable discovery was made by Chandrasekhar, and so we name the unique mass after him. Our expression for it is, from the previous equation,

$$M_{Ch} = \left(\frac{hc}{3G}\right)^{3/2}\left(\frac{1}{\mu m_p}\right)^2. \qquad (12.18)$$

This evaluates to about 1.4 solar masses. Of course, our calculation is only approximate, but it turns out that we have got the right value almost exactly.

Since real electrons don't exactly obey Equation 12.15, but come closer and closer to it the more relativistic they get, we should regard the Chandrasekhar mass also not as the exact mass of any particular white dwarf but rather as an upper bound on the mass of all white dwarfs. Less massive stars have fewer relativistic electrons. More massive stars simply cannot be supported by electron degeneracy pressure at all.

Is the fact that we have not determined the radius of this star a worry? Not really: again, in a real star, the electrons are not fully relativistic, the electron gas is not perfectly ideal, and there is some pressure support from the protons or other nuclei. All these make small corrections, but they are enough to guarantee that any real star's radius will be determined by its mass. The radius will be about the radius of the Earth, as before.

Importantly, the equation of state of the relativistic white dwarf is also a polytrope, but this time with a different power. Steps similar to those used in Investigation 12.3 on page 143 give the relation

$$p = \beta\rho^{4/3}, \qquad (12.19)$$

so that the polytropic index is 3.

Now we remind ourselves of the calculation we did for the stability of the Sun, in Investigation 8.8 on page 101. A polytrope of index 3 is only marginally stable against collapse. Any small correction to the properties of white dwarfs could cause them to be unstable. One correction is that, on compression, some electrons and protons tend to combine into neutrons, removing electrons from the degenerate sea and reducing its pressure. This makes collapse more likely. A second correction is general relativity: in Einstein's theory of gravity, the critical polytropic index actually needs to be somewhat smaller than 3, so that an $n = 3$ gas is actually unstable. Both of these effects become important for highly relativistic white dwarfs, and lead them to be unstable to gravitational collapse a bit before they reach the Chandrasekhar mass.

It is interesting to note that the Chandrasekhar mass can be expressed in terms of two simpler masses: the proton mass m_p and a number with the dimensions of mass that is built only out of the fundamental constants of physics, h, c, and G:

$$m_{Pl} = \left(\frac{hc}{G}\right)^{1/2} = 5.5 \times 10^{-8}\,\text{kg}. \qquad (12.20)$$

This mass is called the **Planck mass**, hence the symbol m_{Pl}. In terms of these simple masses we have

$$M_{Ch} = \left(\frac{1}{3^{3/2}\mu^2}\right)\frac{m_{Pl}^3}{m_p^2}. \qquad (12.21)$$

The Planck mass was first discussed by Planck himself. He noticed, soon after introducing his constant h, that from the fundamental constants h, c, and G one could build numbers with any dimensions one wanted: a mass, a length, a time, and so on. These are now called the Planck mass, the Planck length, etc. We do not yet know exactly what role these quantities play in physics, but we expect it to be fundamentally associated with the quantization of gravity, since they involve both h and G. We will return to this in Chapter 21.

Exercise 12.5.1: ***Deriving the Chandrasekhar mass***

Derive the expression in Equation 12.18 by the indicated method.

Exercise 12.5.2: ***Relativistic degenerate gas equation of state***

Find the constant β in Equation 12.19.

different. Since stellar death provided our Solar System with the raw ingredients of life, we owe much to the Chandrasekhar mass!

Neutron stars

In this section: neutron stars are also supported by degeneracy pressure, but here it is the neutrons which form the supporting distribution. Their maximum mass exceeds $2M_\odot$.

What, then, happens to a contracting star if its mass exceeds the Chandrasekhar mass? Electron degeneracy fails because the electrons have become relativistic, but the protons are still available. Because the proton mass is much larger than that of the electron, protons do not become relativistic until the star is much smaller. When the star is the size of a white dwarf the proton degeneracy pressure is negligible, but as the star contracts further this pressure grows until it can support the star.

The calculations in Investigation 12.4 on page 144 show that a degenerate star's